

# A NOTE FOR GROMOV'S DISTANCE FUNCTIONS ON THE SPACE OF MM-SPACES

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ABSTRACT. This is just a note for [1, Chapter 3 $\frac{1}{2}_+$ ]. Maybe this note is obvious for a reader who knows metric geometry. I wish that someone study further in this direction.

Comments and questions are welcome.

## 1. THE BOX DISTANCE FUNCTION

**Definition 1.1.** Let  $\lambda \geq 0$  and  $(X, \mu)$  be a measure space with  $\mu(X) < +\infty$ . For two maps  $d_1, d_2 : X \times X \rightarrow \mathbb{R}$ , we define a number  $\square_\lambda(d_1, d_2)$  as the infimum of  $\varepsilon > 0$  such that there exists a measurable subset  $T_\varepsilon \subseteq X$  of measure at least  $\mu(X) - \lambda\varepsilon$  satisfying  $|d_1(x, y) - d_2(x, y)| \leq \varepsilon$  for any  $x, y \in T_\varepsilon$ .

It is easy to see that this is a distance function on the set of all functions on  $X \times X$ , and the two distance functions  $\square_\lambda$  and  $\square_{\lambda'}$  are equivalent to each other for any  $\lambda, \lambda' > 0$ .

An *mm-space* is a triple  $(X, d_X, \mu_X)$ , where  $d_X$  is a complete separable metric on a set  $X$  and  $\mu_X$  a finite Borel measure on  $(X, d_X)$ . Two mm-spaces are *isomorphic* to each other if there is a measure preserving isometry between the supports of their measures. We denote by  $\mathcal{L}$  the Lebesgue measure on  $\mathbb{R}$ .

**Definition 1.2** (parameter). Let  $X$  be an mm-space and  $\mu_X(X) = m$ . Then, there exists a Borel measurable map  $\varphi : [0, m] \rightarrow X$  with  $\varphi_*(\mathcal{L}) = \mu_X$ , where  $\varphi_*(\mathcal{L})$  stands for the push-forward measure of  $\mathcal{L}$  by  $\varphi$ . We call  $\varphi$  a *parameter* of  $X$ .

Note that if the support of  $X$  is not a one-point, then its parameter is not unique.

**Definition 1.3** (Gromov's box distance function). If two mm-spaces  $X, Y$  satisfy  $\mu_X(X) = \mu_Y(Y) = m$ , we define

$$\square_\lambda(X, Y) := \inf \square_\lambda(\varphi_X^* d_X, \varphi_Y^* d_Y),$$

where the infimum is taken over all parameters  $\varphi_X : [0, m] \rightarrow X$ ,  $\varphi_Y : [0, m] \rightarrow Y$ , and  $\varphi_X^* d_X$  is defined by  $\varphi_X^* d_X(s, t) := d_X(\varphi_X(s), \varphi_X(t))$  for  $s, t \in [0, m]$ . If  $\mu_X(X) < \mu_Y(Y)$ , putting  $m := \mu_X(X)$ ,  $m' := \mu_Y(Y)$ , we define

$$\square_\lambda(X, Y) := \square_\lambda\left(X, \frac{m}{m'}Y\right) + m' - m,$$

where  $(m/m')Y := (Y, d_Y, (m/m')\mu_Y)$ .

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We denote by  $\mathcal{X}$  the space of all isomorphic class of mm-spaces.  $\square_\lambda$  is a distance function on  $\mathcal{X}$  for any  $\lambda \geq 0$  (See Theorem 1.10). Note that the distances  $\square_\lambda$  and  $\square_{\lambda'}$  are equivalent to each other for distinct  $\lambda, \lambda' > 0$ .

The following two lemmas are easy to prove, so we omit the proof.

**Lemma 1.4.** *Assume that two mm-spaces  $X, Y$  satisfy  $m := \mu_X(X) = \mu_Y(Y)$  and a Borel measurable map  $\Phi : [0, m] \rightarrow [0, m]$  satisfies  $\Phi_*(\mathcal{L}) = \mathcal{L}$ . Then, both  $\varphi_X \circ \Phi : [0, m] \rightarrow X$  and  $\varphi_Y \circ \Phi : [0, m] \rightarrow Y$  are parameters, and the inequality*

$$\square_\lambda((\varphi_X \circ \Phi)^* d_X, (\varphi_Y \circ \Phi)^* d_Y) \leq \square_\lambda(\varphi_X^* d_X, \varphi_Y^* d_Y)$$

holds.

**Lemma 1.5.** *Assume that two mm-spaces  $X, Y$  satisfy  $m := \mu_X(X) = \mu_Y(Y)$  and let  $0 < \alpha \leq 1$ . Then, we have*

$$\alpha \square_\lambda(X, Y) \leq \square_\lambda(\alpha X, \alpha Y) \leq \square_\lambda(X, Y).$$

The following lemma is the key to prove the triangle inequality for  $\square_\lambda$ .

**Lemma 1.6.** *Let  $(X, d_X, \mu_X)$  be a mm-space and  $\varphi_X : [0, m] \rightarrow X, \psi_X : [0, m] \rightarrow X$  be two parameters. Then, for any  $\varepsilon > 0$ , there exist two Borel measurable maps  $\Phi_1, \Phi_2 : [0, m] \rightarrow [0, m]$  such that  $\Phi_{1*}(\mathcal{L}) = \mathcal{L}, \Phi_{2*}(\mathcal{L}) = \mathcal{L}$ , and*

$$\square_0((\varphi_X \circ \Phi_1)^* d_X, (\psi_X \circ \Phi_2)^* d_X) < \varepsilon.$$

*Proof.* To prove the lemma, we shall approximate  $X$  by a countable space. For any  $\varepsilon > 0$ , there exists a sequence  $\{X_i\}_{i=1}^\infty$  of pairwise disjoint Borel subsets of  $X$  such that  $X = \bigcup_{i=1}^\infty X_i$  and  $\text{diam } X_i < \varepsilon$  for each  $i \in \mathbb{N}$ . Fix a point  $x_i \in X_i$  for each  $i \in \mathbb{N}$ . We define a distance between  $x_i$  and  $x_j$  by  $d_{X'}(x_i, x_j) := d_X(x_i, x_j)$ , and a Borel measure  $\mu_{X'}$  on  $X'$  by  $\mu_{X'}(\{x_i\}) := \mu_X(X_i)$ . Define two maps  $\varphi_{X'} : [0, m] \rightarrow X'$  and  $\psi_{X'} : [0, m] \rightarrow X'$  by  $\varphi_{X'}(t) := x_i$  for  $t \in \varphi_X^{-1}(X_i)$  and  $\psi_{X'}(t) := x_i$  for  $t \in \psi_X^{-1}(X_i)$ . It is easy to see that  $\square_0(\varphi_X^* d_X, \varphi_{X'}^* d_{X'}) < 2\varepsilon$  and  $\square_0(\psi_X^* d_X, \psi_{X'}^* d_{X'}) < 2\varepsilon$ . Put  $\Phi_{X'}(t) := x_1$  for  $t \in [0, \mu_{X'}(\{x_1\})]$ , and  $\Phi_{X'}(t) := x_i$  for  $t \in \left[ \sum_{k=1}^{i-1} \mu_{X'}(\{x_k\}), \sum_{k=1}^i \mu_{X'}(\{x_k\}) \right)$ ,  $i = 2, 3, \dots, n$ .

We construct a Borel measurable map  $\Phi_1^1 : [0, \mu_{X'}(\{x_1\})] \rightarrow \varphi_{X'}^{-1}(\{x_1\})$  as follows: There is a sequence  $\{K_n\}_{n=1}^\infty$  of compact subsets of  $\varphi_{X'}^{-1}(\{x_1\})$  such that  $K_1 \subseteq K_2 \subseteq \dots$  and  $\mathcal{L}(K_n) \rightarrow \mathcal{L}(\varphi_{X'}^{-1}(\{x_1\}))$ . Take a Borel measurable map  $\Phi_1^{11} : [0, \mathcal{L}(K_1)) \rightarrow K_1$  such that  $(\Phi_1^{11})_*(\mathcal{L}) = \mathcal{L}$ . For each  $i = 2, 3, \dots$ , we find a sequence  $\{(a_n^i, b_n^i)\}_{n=1}^\infty$  of pairwise disjoint open intervals such that  $K_i \setminus K_{i-1} = K_i \cap \bigcup_{k=1}^\infty (a_k^i, b_k^i)$ . Take Borel measurable maps  $\Psi_1 : I_1 := [\mathcal{L}(K_{i-1}), \mathcal{L}(K_{i-1}) + \mathcal{L}(K_i \cap [a_1^i, b_1^i])] \rightarrow K_i \cap [a_1^i, b_1^i]$  and  $\Psi_k : I_k := [\mathcal{L}(K_{i-1}) + \sum_{l=1}^{k-1} \mathcal{L}(K_i \cap [a_l^i, b_l^i]), \mathcal{L}(K_{i-1}) + \sum_{l=1}^k \mathcal{L}(K_i \cap [a_l^i, b_l^i])] \rightarrow K_i \cap [a_k^i, b_k^i]$ ,  $k = 2, 3, \dots$ , such that  $(\Psi_k)_*(\mathcal{L}) = \mathcal{L}$  for  $k = 1, 2, \dots$ . By modifying each  $\Psi_k$ , we

may assume that  $\Psi_k(I_k) \subseteq K_i \cap (a_k^i, b_k^i)$ . Then we define a Borel measurable map  $\Phi_1^{1^i} : [\mathcal{L}(K_{i-1}), \mathcal{L}(K_i)) \rightarrow K_i \setminus K_{i-1}$  by  $\Phi_1^{1^i}(t) := \Psi_k(t)$  if  $t \in I_k$ . Put  $\Phi_1^1(t) := \Phi_1^{1^1}(t)$  for  $t \in [0, \mathcal{L}(K_1))$  and  $\Phi_1^1(t) := \Phi_1^{1^i}(t)$  for  $t \in [\mathcal{L}(K_{i-1}), \mathcal{L}(K_i))$ . It is obvious that this map  $\Phi_1^1$  satisfies  $(\Phi_1^1)_*(\mathcal{L}) = \mathcal{L}$ . In this way, we find a sequence of Borel measurable maps  $\left\{ \Phi_1^i : \left[ \sum_{k=1}^{i-1} \mu_{X'}(\{x_k\}), \sum_{k=1}^i \mu_{X'}(\{x_k\}) \right) \rightarrow \varphi_{X'}^{-1}(\{x_i\}) \right\}_{i=2}^\infty$  such that  $(\Phi_1^i)_*(\mathcal{L}) = \mathcal{L}$  for each  $i = 2, 3, \dots$ .

Define a Borel measurable map  $\Phi_1 : [0, m) \rightarrow [0, m)$  by  $\Phi_1(t) := \Phi_1^1(t)$  for  $t \in [0, \mu_{X'}(\{x_1\})$  and  $\Phi_1(t) := \Phi_1^i(t)$  for  $t \in \left[ \sum_{k=1}^{i-1} \mu_{X'}(\{x_k\}), \sum_{k=1}^i \mu_{X'}(\{x_k\}) \right)$ ,  $i = 2, 3, \dots$ . From the above construction, it follows that  $\Phi_{1*}(\mathcal{L}) = \mathcal{L}$  and  $\Phi_{X'} = \varphi_{X'} \circ \Phi_1$ . In the same way, we find a Borel measurable map  $\Phi_2 : [0, m) \rightarrow [0, m)$  such that  $\Phi_{2*}\mathcal{L} = \mathcal{L}$  and  $\Phi_{X'} = \psi_{X'} \circ \Phi_2$ . Therefore, by using Lemma 1.4, we have

$$\begin{aligned} \square_0((\varphi_{X'} \circ \Phi_1)^* d_X, (\psi_X \circ \Phi_2)^* d_X) &\leq \square_0((\varphi_X \circ \Phi_1)^* d_X, (\varphi_{X'} \circ \Phi_1)^* d_{X'}) \\ &\quad + \square_0((\psi_{X'} \circ \Phi_2)^* d_{X'}, (\psi_X \circ \Phi_2)^* d_X) \\ &\leq \square_0(\varphi_X^* d_X, \varphi_{X'}^* d_{X'}) \\ &\quad + \square_0(\psi_{X'}^* d_{X'}, \psi_X^* d_X) \\ &< 4\varepsilon. \end{aligned}$$

This completes the proof.  $\square$

**Lemma 1.7.** *For any  $\lambda \geq 0$ ,  $\square_\lambda$  satisfies the triangle inequality.*

*Proof.* Let  $(X, d_X, \mu_X), (Y, d_Y, \mu_Y), (Z, d_Z, \mu_Z)$  be mm-spaces and put  $m := \mu_X(X), m' := \mu_Y(Y), m'' := \mu_Z(Z)$ .

**Case 1.**  $m = m' = m''$ .

Let  $\varphi_X : [0, m] \rightarrow X, \varphi_Y : [0, m] \rightarrow Y, \psi_Y : [0, m] \rightarrow Y, \varphi_Z : [0, m] \rightarrow Z$  be any parameters. By virtue of Lemma 1.6, for any  $\varepsilon > 0$ , there exists two Borel measurable maps  $\Phi_1 : [0, m] \rightarrow [0, m], \Phi_2 : [0, m] \rightarrow [0, m]$  such that  $\Phi_{1*}(\mathcal{L}) = \mathcal{L}, \Phi_{2*}(\mathcal{L}) = \mathcal{L}$ , and

$$\square_\lambda((\varphi_Y \circ \Phi_1)^* d_Y, (\psi_Y \circ \Phi_2)^* d_Y) \leq \square_0((\varphi_Y \circ \Phi_1)^* d_Y, (\psi_Y \circ \Phi_2)^* d_Y) < \varepsilon.$$

Applying Lemma 1.4, we get

$$\begin{aligned} &\square_\lambda(\varphi_X^* d_X, \varphi_Y^* d_Y) + \square_\lambda(\psi_Y^* d_Y, \varphi_Z^* d_Z) \\ &\geq \square_\lambda((\varphi_X \circ \Phi_1)^* d_X, (\varphi_Y \circ \Phi_1)^* d_Y) + \square_\lambda((\psi_Y \circ \Phi_2)^* d_Y, (\varphi_Z \circ \Phi_2)^* d_Z) \\ &\geq \square_\lambda((\varphi_X \circ \Phi_1)^* d_X, (\varphi_Z \circ \Phi_2)^* d_Z) - \square_\lambda((\varphi_Y \circ \Phi_1)^* d_Y, (\psi_Y \circ \Phi_2)^* d_Y) \\ &\geq \square_\lambda(X, Z) - \varepsilon, \end{aligned}$$

which shows  $\square_\lambda(X, Y) + \square_\lambda(Y, Z) \geq \square_\lambda(X, Z) - \varepsilon$ .

**Case 2.**  $m \neq m', m = m'$ .

If  $m < m'$ , by Lemma 1.5, we have

$$\begin{aligned}
\sqsubseteq_\lambda(X, Y) + \sqsubseteq_\lambda(Y, Z) &= \sqsubseteq_\lambda\left(X, \frac{m}{m'}Y\right) + \sqsubseteq_\lambda(Y, Z) + m' - m \\
&\geq \sqsubseteq_\lambda\left(X, \frac{m}{m'}Y\right) + \sqsubseteq_\lambda\left(\frac{m}{m'}Y, \frac{m}{m'}Z\right) + m' - m \\
&\geq \sqsubseteq_\lambda\left(X, \frac{m}{m'}Z\right) + m' - m \\
&= \sqsubseteq_\lambda(X, Z).
\end{aligned}$$

If  $m > m'$ , we have

$$\begin{aligned}
\sqsubseteq_\lambda(X, Z) + \sqsubseteq_\lambda(Y, Z) &= \sqsubseteq_\lambda\left(\frac{m'}{m}X, Y\right) + \sqsubseteq_\lambda(Y, Z) + m - m' \\
&\geq \sqsubseteq_\lambda\left(\frac{m'}{m}X, Z\right) + m - m' \\
&= \sqsubseteq_\lambda(X, Z).
\end{aligned}$$

**Case 3.**  $m \neq m', m' \neq m'', m = m''$ .

If  $m < m'$ , we have

$$\sqsubseteq_\lambda(X, Y) + \sqsubseteq_\lambda(Y, Z) = \sqsubseteq_\lambda\left(X, \frac{m}{m'}Y\right) + \sqsubseteq_\lambda\left(\frac{m}{m'}Y, Z\right) + 2(m - m') \geq \sqsubseteq_\lambda(X, Z).$$

If  $m > m'$ , applying Lemma 1.5, we get

$$\begin{aligned}
\sqsubseteq_\lambda(X, Y) + \sqsubseteq_\lambda(Y, Z) &= \sqsubseteq_\lambda\left(\frac{m'}{m}X, Y\right) + \sqsubseteq_\lambda\left(Y, \frac{m'}{m}Z\right) + 2(m - m') \\
&\geq \sqsubseteq_\lambda\left(\frac{m'}{m}X, \frac{m'}{m}Z\right) + 2(m - m') \\
&\geq \frac{m'}{m} \sqsubseteq_\lambda(X, Z) + 2(m - m').
\end{aligned}$$

$m \geq \sqsubseteq_\lambda(X, Z)$  directly implies that

$$2(m - m') \geq \left(1 - \frac{m'}{m}\right) \sqsubseteq_\lambda(X, Z).$$

Thus, we obtain  $\sqsubseteq_\lambda(X, Y) + \sqsubseteq_\lambda(Y, Z) \geq \sqsubseteq_\lambda(X, Z)$ .

**Case 4.**  $m \neq m', m \neq m'', m' \neq m''$ .

If  $m < m', m' < m''$ , by using Lemma 1.5, we have

$$\begin{aligned}
\sqsubseteq_\lambda(X, Y) + \sqsubseteq_\lambda(Y, Z) &= \sqsubseteq_\lambda\left(X, \frac{m}{m'}Y\right) + m' - m + \sqsubseteq_\lambda\left(Y, \frac{m'}{m''}Z\right) + m'' - m' \\
&\geq \sqsubseteq_\lambda\left(X, \frac{m}{m'}Y\right) + \sqsubseteq_\lambda\left(\frac{m}{m'}Y, \frac{m}{m''}Z\right) + m'' - m \\
&\geq \sqsubseteq_\lambda\left(X, \frac{m}{m''}Z\right) + m'' - m \\
&= \sqsubseteq_\lambda(X, Z).
\end{aligned}$$

If  $m < m', m'' < m', m < m'',$  by Lemma 1.5, we get

$$\begin{aligned}
\sqsubseteq_\lambda(X, Y) + \sqsubseteq_\lambda(Y, Z) &= \sqsubseteq_\lambda\left(X, \frac{m}{m'}Y\right) + m' - m + \sqsubseteq_\lambda\left(\frac{m''}{m'}Y, Z\right) + m' - m'' \\
&= \sqsubseteq_\lambda\left(X, \frac{m}{m'}Y\right) + \sqsubseteq_\lambda\left(\frac{m''}{m'}Y, Z\right) + 2m' - m - m'' \\
&\geq \sqsubseteq_\lambda\left(X, \frac{m}{m'}Y\right) + \sqsubseteq_\lambda\left(\frac{m}{m'}Y, \frac{m}{m''}Z\right) + m'' - m \\
&\geq \sqsubseteq_\lambda\left(X, \frac{m}{m''}Z\right) + m'' - m \\
&= \sqsubseteq_\lambda(X, Z).
\end{aligned}$$

We prove the same way for the case of  $m < m', m'' < m', m'' < m.$  This completes the proof of Lemma 1.6.  $\square$

Let  $X$  be a mm-space and  $M_r$  be the set of all real  $r \times r$  matrices. Then we define a Borel measurable map  $K_r : X^r \rightarrow M_r$  by  $K_r(x_1, \dots, x_r) := (d_X(x_i, x_j))_{i,j},$  and a Borel measure on  $M_r$  by  $\underline{\mu}_r^X := (K_r)_*((\mu_X)^r).$

**Theorem 1.8** (mm-Reconstruction theorem, [1, Section 3 $\frac{1}{2}$ .5, 3 $\frac{1}{2}$ .7]). *If two mm-spaces  $X, X'$  have  $\underline{\mu}_r^X = \underline{\mu}_r^{X'}$  for all  $r \in \mathbb{N}$ , then  $X$  and  $X'$  are isomorphic to each other.*

A. M. Vershik gave the another proof of the reconstruction theorem in [4, Section 2, Theorem]. We also refer to [2, Section 2, Theorem 2.1] for his proof. In [2], T. Kondo generalized the reconstruction theorem to the space of Borel probability measures on  $\mathcal{X}$ .

**Lemma 1.9.** *Let  $(X, d, \mu)$  be a mm-space, and  $\varphi_X : [0, m] \rightarrow X$  be a parameter of  $X$ . We set  $S := ([0, m], \varphi_X^* d, \mathcal{L})$ . Then, we have  $\underline{\mu}_r^X = \underline{\mu}_r^S$  for all  $r = 1, 2, \dots$ .*

*Proof.* Let  $\varphi : [0, m]^r \rightarrow X^r$  be a Borel measurable map defined by  $\varphi(t_1, \dots, t_r) := (\varphi_X(t_1), \dots, \varphi_X(t_r)).$  Obviously,  $\varphi_*(\mathcal{L}^r) = (\mu_X)^r.$  Therefore, for any Borel subset  $A \subseteq M_r,$  we obtain

$$\begin{aligned}
\underline{\mu}_r^S(A) &= \mathcal{L}^r(\{(t_1, \dots, t_r) \in [0, m]^r \mid (\varphi_X^* d_X(t_i, t_j))_{i,j} \in A\}) \\
&= \varphi_*(\mathcal{L}^r)(\{(x_1, \dots, x_r) \in X^r \mid (d_X(x_i, x_j))_{i,j} \in A\}) \\
&= (\mu_X)^r(\{(x_1, \dots, x_r) \in X^r \mid (d_X(x_i, x_j))_{i,j} \in A\}) \\
&= \underline{\mu}_r^X(A).
\end{aligned}$$

This completes the proof.  $\square$

**Theorem 1.10** (Gromov, cf. [1, Section 3 $\frac{1}{2}$ .6 Corollary]). *For any  $\lambda \geq 0,$   $\sqsubseteq_\lambda$  is a distance function on  $\mathcal{X}.$*

*Proof.* Since  $\sqsubseteq_\lambda$  satisfies the triangle inequality, we only prove that  $\sqsubseteq_\lambda(X, Y) = 0$  implies  $X \cong Y.$  Supposing that  $\sqsubseteq_\lambda(X, Y) = 0,$  we shall show  $\underline{\mu}_r^X = \underline{\mu}_r^Y$  for any  $r \in \mathbb{N}.$  Then, by Theorem 1.8, we get  $X \cong Y.$

Since  $\square_\lambda(X, Y) = 0$ , there exist a sequence  $\{\varphi_{X,n}\}_{n=1}^\infty$  of parameters of  $X$  and a sequence  $\{\varphi_{Y,n}\}_{n=1}^\infty$  of parameters of  $Y$  such that  $\square_\lambda(\varphi_{X,n}^* dX, \varphi_{Y,n}^* dY) \rightarrow 0$  as  $n \rightarrow \infty$ . Hence, there exist a sequence  $\{\varepsilon_n\}_{n=1}^\infty$  of positive numbers and a sequence  $\{Z_n\}_{n=1}^\infty$  of Borel subsets of  $[0, m]$  such that  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ ,  $\mathcal{L}(Z_n) \geq m - \lambda\varepsilon_n$ , and  $|\varphi_{X,n}^* dX(s, t) - \varphi_{Y,n}^* dY(s, t)| \leq \varepsilon_n$  for any  $s, t \in Z_n$ . Let  $U \subseteq M_r$  be an arbitrary open set and denote by  $d_{M_r}$  the usual Euclidean distance on  $M_r$ , that is,

$$d_{M_r}((a_{ij})_{i,j}, (b_{ij})_{i,j}) := \left( \sum_{i,j=1}^r (a_{ij} - b_{ij})^2 \right)^{1/2}.$$

Put

$$\begin{aligned} X_{n,\varepsilon} &:= \{(t_1, \dots, t_r) \in [0, m]^r \mid (\varphi_{X,n}^* dX(t_i, t_j))_{i,j} \in U \setminus (M_r \setminus U)_{+\varepsilon}\}, \\ Y_n &:= \{(t_1, \dots, t_r) \in [0, m]^r \mid (\varphi_{Y,n}^* dY(t_i, t_j))_{i,j} \in U\}. \end{aligned}$$

We take  $n_0 \in \mathbb{N}$  such that  $\varepsilon_n < \varepsilon/r$  for any  $n \geq n_0$ .

**Claim 1.11.** *For any  $n \geq n_0$ , we have  $X_{n,\varepsilon} \subseteq Y_n \cup ([0, m]^r \setminus (Z_n)^r)$ .*

*Proof.* Take any  $(t_1, \dots, t_r) \in X_{n,\varepsilon}$ . If  $(t_1, \dots, t_r) \in (Z_n)^r$ , then for any  $i, j$  we have

$$|\varphi_{X,n}^* dX(t_i, t_j) - \varphi_{Y,n}^* dY(t_i, t_j)| \leq \varepsilon_n < \varepsilon/r,$$

which implies that  $d_{M_r}((\varphi_{X,n}^* dX(t_i, t_j))_{i,j}, (\varphi_{Y,n}^* dY(t_i, t_j))_{i,j}) < \varepsilon$ . Hence, we obtain  $(\varphi_{Y,n}^* dY(t_i, t_j))_{i,j} \in U$ . This completes the proof of the claim.  $\square$

Put  $S_n := ([0, m], \varphi_{X,n}^* dX, \mathcal{L})$  and  $S'_n := ([0, m], \varphi_{Y,n}^* dY, \mathcal{L})$  and let  $m := \mu_X(X) = \mu_Y(Y)$ . Combining Lemma 1.9 and Claim 1.11, for any  $n \geq n_0$  we have

$$\begin{aligned} \underline{\mu}_r^X(U \setminus (M_r \setminus U)_{+\varepsilon}) &= \underline{\mu}_r^{S_n}(U \setminus (M_r \setminus U)_{+\varepsilon}) = \mathcal{L}^r(X_{n,\varepsilon}) \leq \mathcal{L}^r(Y_n \cup ([0, m]^r \setminus (Z_n)^r)) \\ &\leq \mathcal{L}^r(Y_n) + \mathcal{L}^r([0, m]^r \setminus (Z_n)^r) \\ &\leq \underline{\mu}_r^{S'_n}(U) + rm^{r-1}\lambda\varepsilon_n \\ &= \underline{\mu}_r^Y(U) + rm^{r-1}\lambda\varepsilon_n. \end{aligned}$$

In the above inequality, let first  $n \rightarrow \infty$  and next  $\varepsilon \rightarrow 0$ . Then we get  $\underline{\mu}_r^X(U) \leq \underline{\mu}_r^Y(U)$ . The same argument shows that  $\underline{\mu}_r^Y(U) \leq \underline{\mu}_r^X(U)$ , which yields  $\underline{\mu}_r^X(U) = \underline{\mu}_r^Y(U)$ . This completes the proof of Theorem 1.10.  $\square$

## 2. THE OBSERVABLE DISTANCE FUNCTION

For a measure space  $(X, \mu)$  with  $\mu(X) < +\infty$ , we denote by  $\mathcal{F}(X, \mathbb{R})$  the space of all functions on  $X$ . Given  $\lambda \geq 0$  and  $f, g \in \mathcal{F}(X, \mathbb{R})$ , we put

$$\text{me}_\lambda(f, g) := \inf\{\varepsilon > 0 \mid \mu(\{x \in X \mid |f(x) - g(x)| \geq \varepsilon\}) \leq \lambda\varepsilon\}.$$

Note that this  $\text{me}_\lambda$  is a distance function on  $\mathcal{F}(X, \mathbb{R})$  for any  $\lambda \geq 0$  and its topology on  $\mathcal{F}(X, \mathbb{R})$  coincides with the topology of the convergence in measure for any  $\lambda > 0$ . Also, the distance functions  $\text{me}_\lambda$  for all  $\lambda > 0$  are mutually equivalent.

We recall that the *Hausdorff distance* between two closed subsets  $A$  and  $B$  in a metric space  $X$  is defined by

$$d_H(A, B) := \inf\{\varepsilon > 0 \mid A \subseteq B_\varepsilon, B \subseteq A_\varepsilon\},$$

where  $A_\varepsilon$  is a closed  $\varepsilon$ -neighborhood of  $A$ .

Let  $(X, \mu)$  be a measure space with  $\mu(X) < +\infty$ . For a semi-distance function  $d$  on  $X$ , we indicate by  $\mathcal{Lip}_1(d)$  the space of all 1-Lipschitz functions on  $X$  with respect to  $d$ . Note that  $\mathcal{Lip}_1(d)$  is a closed subset in  $(\mathcal{F}(X, \mathbb{R}), \text{me}_\lambda)$  for any  $\lambda \geq 0$ .

**Definition 2.1.** For  $\lambda \geq 0$  and two semi-distance functions  $d, d'$  on  $X$ , we define

$$H_\lambda \mathcal{L}_{t_1}(d, d') := d_H(\mathcal{Lip}_1(d), \mathcal{Lip}_1(d')),$$

where  $d_H$  stands for the Hausdorff distance function in  $(\mathcal{F}(X, \mathbb{R}), \text{me}_\lambda)$ .

This  $H_\lambda \mathcal{L}_{t_1}$  is actually a distance function on the space of all semi-distance functions on  $X$  for all  $\lambda \geq 0$ , and the two distance functions  $H_\lambda \mathcal{L}_{t_1}$  and  $H_{\lambda'} \mathcal{L}_{t_1}$  are equivalent to each other for any  $\lambda, \lambda' > 0$ .

**Lemma 2.2.** For any two semi-distance functions  $d, d'$  on  $X$ , we have

$$H_\lambda \mathcal{L}_{t_1}(d, d') \leq \square_\lambda(d, d').$$

*Proof.* For any  $\varepsilon > 0$  with  $\square_\lambda(X, Y) < \varepsilon$ , there exists a measurable subset  $T_\varepsilon \subseteq X$  such that  $\mu(X \setminus T_\varepsilon) \leq \lambda\varepsilon$  and  $|d(x, y) - d'(x, y)| \leq \varepsilon$  for any  $x, y \in T_\varepsilon$ . Given arbitrary  $f \in \mathcal{Lip}_1(d)$ , we define  $\tilde{f} \in \mathcal{F}(X, \mathbb{R})$  by  $\tilde{f}(x) := \inf\{f(y) + d'(x, y) \mid y \in T_\varepsilon\}$ . We see easily that  $\tilde{f} \in \mathcal{Lip}_1(d')$  and  $\tilde{f}(x) \leq f(x)$  for any  $x \in T_\varepsilon$ . Taking any  $x \in T_\varepsilon$ , we have

$$\begin{aligned} |f(x) - \tilde{f}(x)| &= f(x) - \tilde{f}(x) \\ &= \sup\{f(x) - f(y) - d'(x, y) \mid y \in T_\varepsilon\} \\ &\leq \sup\{d(x, y) - d'(x, y) \mid y \in T_\varepsilon\} \\ &\leq \varepsilon. \end{aligned}$$

Therefore, we get  $\text{me}_\lambda(f, \tilde{f}) \leq \varepsilon$ , which implies  $\mathcal{Lip}_1(d) \subseteq (\mathcal{Lip}_1(d'))_\varepsilon$ . Similarly, we also have  $\mathcal{Lip}_1(d') \subseteq (\mathcal{Lip}_1(d))_\varepsilon$ , which yields  $H_\lambda \mathcal{L}_{t_1}(d, d') \leq \varepsilon$ . This completes the proof.  $\square$

**Definition 2.3** (Observable distance function). If two mm-spaces  $X, Y$  satisfy  $\mu_X(X) = \mu_Y(Y) = m$ , we define

$$\underline{H}_\lambda \mathcal{L}_{t_1}(X, Y) := \inf H_\lambda \mathcal{L}_{t_1}(\varphi_X^* d_X, \varphi_Y^* d_Y),$$

where the infimum is taken over all parameters  $\varphi_X : [0, m] \rightarrow X$ ,  $\varphi_Y : [0, m] \rightarrow Y$ . If  $\mu_X(X) < \mu_Y(Y)$ , putting  $m := \mu_X(X)$ ,  $m' := \mu_Y(Y)$ , we define

$$\underline{H}_\lambda \mathcal{L}_{t_1}(X, Y) := \underline{H}_\lambda \mathcal{L}_{t_1}\left(X, \frac{m}{m'} Y\right) + m' - m.$$

$\underline{H}_\lambda \mathcal{L}_{t_1}$  is a distance function on  $\mathcal{X}$  for any  $\lambda \geq 0$  (See Theorem 2.8). Note that the distance functions  $\underline{H}_\lambda \mathcal{L}_{t_1}$  and  $\underline{H}_{\lambda'} \mathcal{L}_{t_1}$  are equivalent to each other for any  $\lambda, \lambda' > 0$ .

The proofs of following four lemmas are easy.

**Lemma 2.4.** *For any parameter  $\varphi_X : [0, m] \rightarrow X$  of  $X$ , we have*

$$\mathcal{Lip}_1(\varphi_X^* d_X) = \{f \circ \varphi_X \mid f \in \mathcal{Lip}_1(d_X)\}.$$

**Lemma 2.5.** *Assume that two mm-spaces  $X, Y$  satisfy  $m := \mu_X(X) = \mu_Y(Y)$  and a Borel measurable map  $\Phi : [0, m] \rightarrow [0, m]$  satisfies  $\Phi_*(\mathcal{L}) = \mathcal{L}$ . Then, we have*

$$H_\lambda \mathcal{L}_{\iota_1}((\varphi_X \circ \Phi)^* d_X, (\varphi_Y \circ \Phi)^* d_Y) = H_\lambda \mathcal{L}_{\iota_1}(\varphi_X^* d_X, \varphi_Y^* d_Y).$$

**Lemma 2.6.** *Assume that two mm-spaces  $X, Y$  satisfy  $m := \mu_X(X) = \mu_Y(Y)$  and let  $0 < \alpha \leq 1$ . Then, we have*

$$\alpha \underline{H}_\lambda \mathcal{L}_{\iota_1}(X, Y) \leq \underline{H}_\lambda \mathcal{L}_{\iota_1}(\alpha X, \alpha Y) \leq \underline{H}_\lambda \mathcal{L}_{\iota_1}(X, Y).$$

**Lemma 2.7.** *Let  $X$  be a mm-space and  $\varphi_X : [0, m] \rightarrow X, \psi_X : [0, m] \rightarrow X$  be two parameters. Then, for any  $\varepsilon > 0$ , there exist two Borel measurable maps  $\Phi_1, \Phi_2 : [0, m] \rightarrow [0, m]$  such that  $(\Phi_1)_*(\mathcal{L}) = \mathcal{L}$ ,  $(\Phi_2)_*(\mathcal{L}) = \mathcal{L}$ , and*

$$H_0 \mathcal{L}_{\iota_1}((\varphi_X \circ \Phi_1)^* d_X, (\psi_X \circ \Phi_2)^* d_X) < \varepsilon.$$

**Theorem 2.8** (Gromov, cf. [1, Section 3 $\frac{1}{2}$ .45]). *For any  $\lambda \geq 0$ ,  $\underline{H}_\lambda \mathcal{L}_{\iota_1}$  is a distance function on  $\mathcal{X}$ .*

*Proof.* Combining Lemma 2.5, 2.6, and 2.7, we see that  $\underline{H}_\lambda \mathcal{L}_{\iota_1}$  satisfies the triangle inequality in the same way of the proof of Lemma 1.7.

To prove “ $\underline{H}_\lambda \mathcal{L}_{\iota_1}(X, Y) = 0 \Rightarrow X \cong Y$ ”, we shall approximate each  $X$  and  $Y$  by finite spaces. Take an arbitrary  $\varepsilon > 0$ . Then, there exists sequences  $\{X_i\}_{i=1}^\infty, \{Y_j\}_{j=1}^\infty$  of pairwise disjoint Borel subsets of  $X, Y$  such that

- (1)  $X = \bigcup_{i=1}^\infty X_i$  and  $\text{diam } X_i \leq \varepsilon$  for any  $i \in \mathbb{N}$ ,
- (2)  $Y = \bigcup_{j=1}^\infty Y_j$  and  $\text{diam } Y_j \leq \varepsilon$  for any  $j \in \mathbb{N}$ .

Put  $m := \mu_X(X) = \mu_Y(Y)$ . Then, there exists  $m_0 \in \mathbb{N}$  such that

$$m - \varepsilon \leq \mu_X\left(\bigcup_{i=1}^{m_0} X_i\right), m - \varepsilon \leq \mu_Y\left(\bigcup_{j=1}^{m_0} Y_j\right).$$

Since  $\underline{H}_1 \mathcal{L}_{\iota_1}(X, Y) = 0$ , there exist a sequence  $\{\varepsilon_n\}$  of positive numbers and sequences  $\{\varphi_{X,n}\}_{n=1}^\infty, \{\varphi_{Y,n}\}_{n=1}^\infty$  of parameters of  $X, Y$  such that  $H_1 \mathcal{L}_{\iota_1}(\varphi_{X,n}^* d_X, \varphi_{Y,n}^* d_Y) < \varepsilon_n$  and  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ . For each  $i, j = 1, \dots, m_0$ , we fix points  $x_i \in X_i$  and  $y_j \in Y_j$ . Define a function  $g_{ni} : [0, m] \rightarrow \mathbb{R}$  by  $g_{ni}(s) := d_X(\varphi_{X,n}(s), x_i)$  for each  $i = 1, 2, \dots, m_0$ . From Lemma 2.4, we have  $g_{ni} \in \mathcal{Lip}_1(\varphi_{X,n}^* d_X)$ . Hence, there exists  $h_{ni} \in \mathcal{Lip}_1(d_Y)$  such that  $\text{me}_1(g_{ni}, h_{ni} \circ \varphi_{Y,n}) < \varepsilon_n$ . Putting

$$A_{ni} := \{s \in [0, m] \mid |g_{ni}(s) - (h_{ni} \circ \varphi_{Y,n})(s)| < \varepsilon_n\},$$



we get  $\mathcal{L}(A_{ni}) \geq m - \varepsilon_n$ . For each  $j = 1, 2, \dots, m_0$ , we define a function  $\tilde{h}_{nj} : [0, m] \rightarrow \mathbb{R}$  by  $\tilde{h}_{nj}(s) := d_Y(\varphi_{Y,n}(s), y_j)$ . By the same argument as above, there exists  $\tilde{g}_{nj} \in \mathcal{Lip}_1(d_X)$  such that  $\mathcal{L}(B_{nj}) \geq m - \varepsilon_n$ , where

$$B_{nj} := \{s \in [0, m] \mid |\tilde{h}_{nj}(s) - (\tilde{g}_{nj} \circ \varphi_{X,n})(s)| < \varepsilon_n\}.$$

So, putting

$$Z_n := \varphi_{X,n}^{-1}\left(\bigcup_{i=1}^{m_0} X_i\right) \cap \varphi_{Y,n}^{-1}\left(\bigcup_{j=1}^{m_0} Y_j\right) \cap \bigcap_{k=1}^{m_0} A_{nk} \cap \bigcap_{l=1}^{m_0} B_{nl},$$

we obtain  $\mathcal{L}(Z_n) \geq 2\varepsilon + 2m_0\varepsilon_n$ .

For any  $s, t \in Z_n$ , there exist  $1 \leq i_1, j_1, i_2, j_2 \leq m_0$  such that

$$\begin{aligned} s &\in \varphi_{X,n}^{-1}(X_{i_1}) \cap \varphi_{Y,n}^{-1}(Y_{j_1}) \cap \bigcap_{k=1}^{m_0} A_{nk} \cap \bigcap_{l=1}^{m_0} B_{nl} \\ \text{and } t &\in \varphi_{X,n}^{-1}(X_{i_2}) \cap \varphi_{Y,n}^{-1}(Y_{j_2}) \cap \bigcap_{k=1}^{m_0} A_{nk} \cap \bigcap_{l=1}^{m_0} B_{nl}. \end{aligned}$$

Since  $t \in \varphi_{X,n}^{-1}(X_{i_2})$  and  $\text{diam } X_{i_2} \leq \varepsilon$ , we have

$$\begin{aligned} d_X(\varphi_{X,n}(s), \varphi_{X,n}(t)) &\leq d_X(\varphi_{X,n}(s), x_{i_2}) + d_X(x_{i_2}, \varphi_{X,n}(t)) \\ &\leq d_X(\varphi_{X,n}(s), x_{i_2}) + \varepsilon. \end{aligned}$$

We also get  $d_X(\varphi_{X,n}(s), x_{i_2}) \leq (h_{ni_2} \circ \varphi_{Y,n})(s) + \varepsilon_n$  by  $s \in \bigcap_{k=1}^{m_0} A_{nk} \subseteq A_{ni_2}$ . Therefore, we obtain

$$\begin{aligned} d_X(\varphi_{X,n}(s), \varphi_{X,n}(t)) &\leq (h_{ni_2} \circ \varphi_{Y,n})(s) + \varepsilon_n + \varepsilon \\ &\leq |(h_{ni_2} \circ \varphi_{Y,n})(s) - (h_{ni_2} \circ \varphi_{Y,n})(t)| + |(h_{ni_2} \circ \varphi_{Y,n})(t)| \\ &\quad + \varepsilon_n + \varepsilon \\ &\leq d_Y(\varphi_{Y,n}(s), \varphi_{Y,n}(t)) + |(h_{ni_2} \circ \varphi_{Y,n})(t)| + \varepsilon_n + \varepsilon. \end{aligned}$$

Since  $t \in \bigcap_{k=1}^{m_0} A_{nk} \cap \varphi_{X,n}^{-1}(X_{i_2})$  and  $\text{diam } X_{i_2} \leq \varepsilon$ , we have  $g_{ni_2}(t) \leq \varepsilon$  and  $|g_{ni_2}(t) - (h_{ni_2} \circ \varphi_{Y,n})(t)| < \varepsilon_n$ , and thus  $|(h_{ni_2} \circ \varphi_{Y,n})(t)| < \varepsilon_n + \varepsilon$ . Therefore, we obtain

$$d_X(\varphi_{X,n}(s), \varphi_{X,n}(t)) \leq d_Y(\varphi_{Y,n}(s), \varphi_{Y,n}(t)) + 2\varepsilon_n + 2\varepsilon.$$

A similar argument shows that

$$d_Y(\varphi_{Y,n}(s), \varphi_{Y,n}(t)) \leq d_X(\varphi_{X,n}(s), \varphi_{X,n}(t)) + 2\varepsilon_n + 2\varepsilon.$$

Hence, we get

$$|d_X(\varphi_{X,n}(s), \varphi_{X,n}(t)) - d_Y(\varphi_{Y,n}(s), \varphi_{Y,n}(t))| \leq 2\varepsilon_n + 2\varepsilon.$$

Therefore, we obtain

$$\square_1(X, Y) \leq \square_1(\varphi_{X,n}^* d_X, \varphi_{Y,n}^* d_Y) \leq 2\varepsilon + 2m_0\varepsilon_n.$$

So, we get  $\square_1(X, Y) = 0$  and  $X \cong Y$ . This completes the proof.  $\square$

Modifying the proof of Theorem 2.8, we get the following corollary:

**Corollary 2.9.** *For any two mm-spaces  $X$  and  $Y$ , we have*

$$\underline{H}_0\mathcal{L}_{l_1}(X, Y) \leq \square_0(X, Y) \leq 2\underline{H}_0\mathcal{L}_{l_1}(X, Y).$$

We also refer to [3, Section 7.4].

### 3. ANOTHER NATURAL METHOD

Let  $\lambda \geq 0$  and  $\varepsilon > 0$ . A map from an mm-space to a metric space, say  $f : X \rightarrow Y$  is called  $\lambda$ -Lipschitz up to  $\varepsilon$  if

$$d_Y(f(x), f(x')) \leq \lambda d_X(x, x') + \varepsilon$$

for all  $x, x'$  in a Borel subset  $X_0 \subseteq X$  with  $\mu_X(X \setminus X_0) \leq \varepsilon$ .

**Proposition 3.1** (cf. [1, Section 3 $\frac{1}{2}$ .15, (3 $_b$ )]). *Let  $(X, d_X, \mu_X)$ ,  $(Y, d_Y, \mu_Y)$  be mm-spaces and  $\lambda \geq 0$ . Let  $\varepsilon_n > 0$  and  $f_n : X \rightarrow Y$  a  $\lambda$ -Lipschitz up to  $\varepsilon_n$  Borel measurable map and assume that  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$  and the sequence  $\{(f_n)_*(\mu_X)\}_{n=1}^\infty$  converges weakly to  $\mu_Y$ . Then, the sequence  $\{f_n\}_{n=1}^\infty$  has a  $\text{me}_1$ -convergent subsequence.*

*Proof.* Without loss of generality, we may assume that  $X = \text{Supp } \mu_X$  and  $\mu_X(X) = \mu_Y(Y) = 1$ .

By choosing a subsequence, we have  $\sum_{n=1}^\infty \varepsilon_n < +\infty$ . From the assumption, there exists a Borel subset  $X_n \subseteq X$  such that  $\mu_X(X \setminus X_n) \leq \varepsilon_n$  and  $d_Y(f_n(x), f_n(y)) \leq \lambda d_X(x, y) + \varepsilon_n$  for any  $x, y \in X_n$ . Put  $X_0 := \bigcup_{n=1}^\infty \bigcap_{i=n}^\infty X_i$ . Since

$$\mu_X(X \setminus X_0) \leq \sum_{i=n}^\infty \mu_X(X \setminus X_i) \leq \sum_{i=n}^\infty \varepsilon_i \rightarrow 0 \text{ as } n \rightarrow \infty,$$

we have  $\mu_X(X_0) = 1$ . Take a countable dense subset  $\{p_j\}_{j=1}^\infty \subseteq X_0$ .

**Claim 3.2.** *The sequence  $\{f_n(p_1)\}_{n=1}^\infty$  has a convergent subsequence.*

*Proof.* The proof is by contradiction. If the sequence  $\{f_n(p_1)\}_{n=1}^\infty$  has no convergent subsequence, then the subset  $A := \{f_1(p_1), f_2(p_1), \dots\}$  is a closed subset in  $Y$ , especially,  $A$  is complete. From the assumption, this set  $A$  is not compact. Hence,  $A$  is not totally bounded, that is, there exists  $\delta > 0$  such that  $A$  has no finite  $2\delta$ -net. Therefore, by

choosing a subsequence, we get  $B_Y(f_j(p_1), \delta) \cap B_Y(f_k(p_1), \delta) = \emptyset$  for any  $j, k$  with  $j \neq k$ . Take  $\delta' > 0$  such that  $0 < \delta' < \delta$  and  $\mu_Y(\partial B_Y(f_j(p_1), \delta')) = 0$  for any  $j \in \mathbb{N}$ . Since

$$\mu_Y\left(\partial\left(\bigcup_{j=1}^{\infty} B_Y(f_j(p_1), \delta')\right)\right) \leq \mu_Y\left(\bigcup_{j=1}^{\infty} \partial B_Y(f_j(p_1), \delta')\right) = 0$$

and  $\{(f_n)_*(\mu_X)\}_{n=1}^{\infty}$  converges weakly to  $\mu_Y$ , we have

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} \mu_X\left(f_n^{-1}(B_Y(f_j(p_1), \delta'))\right) = \sum_{j=1}^{\infty} \mu_Y(B_Y(f_j(p_1), \delta'))$$

and

$$\lim_{n \rightarrow \infty} \mu_X\left(f_n^{-1}(B_Y(f_j(p_1), \delta'))\right) = \mu_Y(B_Y(f_j(p_1), \delta'))$$

for any  $j$ . For any  $\varepsilon > 0$ , there exists  $k_0 \in \mathbb{N}$  such that

$$\sum_{j=1}^{k_0} \mu_Y(B_Y(f_j(p_1), \delta')) + \varepsilon > \sum_{j=1}^{\infty} \mu_Y(B_Y(f_j(p_1), \delta')).$$

Take  $n_0 \in \mathbb{N}$  such that

$$\left| \sum_{j=1}^{k_0} \mu_Y(B_Y(f_j(p_1), \delta')) - \sum_{j=1}^{k_0} \mu_X\left(f_n^{-1}(B_Y(f_j(p_1), \delta'))\right) \right| < \varepsilon$$

and

$$\left| \sum_{j=1}^{\infty} \mu_Y(B_Y(f_j(p_1), \delta')) - \sum_{j=1}^{\infty} \mu_X\left(f_n^{-1}(B_Y(f_j(p_1), \delta'))\right) \right| < \varepsilon$$

for any  $n \geq n_0$ . Hence, for any  $n \geq n_0$  we have

$$\sum_{j=1}^{k_0} \mu_X\left(f_n^{-1}(B_Y(f_j(p_1), \delta'))\right) + 3\varepsilon > \sum_{j=1}^{\infty} \mu_X\left(f_n^{-1}(B_Y(f_j(p_1), \delta'))\right),$$

which implies that

$$\mu_X\left(f_n^{-1}(B_Y(f_n(p_1), \delta'))\right) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Fix  $\delta'' > 0$  with  $\delta'' < \delta'$ . Since  $p_1 \in X_0$ , we get  $d_Y(f_n(p_1), f_n(q)) \leq \lambda d_X(p_1, q) + \varepsilon_n$  for any  $q \in X_n$  and for any sufficiently large  $n \in \mathbb{N}$ . Therefore, we get

$$B_X\left(p_1, \frac{\delta''}{\lambda}\right) \cap X_n \subseteq f_n^{-1}(B_Y(f_n(p_1), \delta'))$$

for any sufficiently large  $n \in \mathbb{N}$ . Hence, we obtain

$$\mu_X\left(B_X\left(p_1, \frac{\delta''}{\lambda}\right) \cap X_n\right) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

which yields  $\mu_X(B_X(p_1, \delta''/\lambda)) = 0$ . This is a contradiction, since  $p_1 \in X = \text{Supp } \mu$ . This completes the proof of the claim.  $\square$

By virtue of Claim 3.2 and the diagonal argument, we have that  $\{f_n(p_j)\}_{n=1}^\infty$  is convergent sequence in  $Y$  for each  $j \in \mathbb{N}$ . We put  $f(p_j) := \lim_{n \rightarrow \infty} f_n(p_j)$  for any  $j \in \mathbb{N}$ . Extend the map  $f : \{p_1, p_2, \dots\} \rightarrow Y$  to  $\tilde{f} : X_0 \rightarrow Y$ , by using  $f$  is a  $\lambda$ -Lipschitz map.

**Claim 3.3.** *For any  $\varepsilon > 0$ , we have  $\mu_X(\{x \in X \mid d_Y(f_n(x), \tilde{f}(x)) \geq \varepsilon\}) \rightarrow 0$  as  $n \rightarrow \infty$ .*

*Proof.* Since  $X_0 \subseteq \bigcup_{j=1}^\infty B_X(p_j, \varepsilon/2)$ , for any  $\delta > 0$  there exists  $k_0 \in \mathbb{N}$  such that

$$\mu_X\left(\bigcup_{j=1}^{k_0} B_X(p_j, \varepsilon/2) \cap X_0\right) \geq 1 - \delta.$$

From the definition, there exists  $n_0 \in \mathbb{N}$  such that  $d_Y(f_n(p_j), \tilde{f}(p_j)) \leq \varepsilon/3$  for any  $n \geq n_0$  and  $j = 1, 2, \dots, k_0$ . Take any  $x \in \bigcup_{j=1}^{k_0} B_X(p_j, \varepsilon/2) \cap X_0$ . There exists  $1 \leq j \leq k_0$  such that  $x \in B_X(p_j, \varepsilon/2)$ . Hence, for any  $n \geq n_0$  we have

$$\begin{aligned} d_Y(f_n(x), \tilde{f}(x)) &\leq d_Y(f_n(x), f_n(p_j)) + d_Y(f_n(p_j), \tilde{f}(p_j)) + d_Y(\tilde{f}(p_j), \tilde{f}(x)) \\ &< (\lambda d_X(x, p_j) + \varepsilon_n) + \frac{\varepsilon}{3} + \lambda \frac{\varepsilon}{2} \\ &\leq \varepsilon_n + \frac{\varepsilon}{3} + \lambda \varepsilon. \end{aligned}$$

Therefore, for any sufficiently large  $n \in \mathbb{N}$ , we obtain

$$\mu_X\left(\left\{x \in X \mid d_Y(f_n(x), \tilde{f}(x)) > \lambda \varepsilon + \frac{\varepsilon}{2}\right\}\right) \leq \mu_X\left(X \setminus \bigcup_{j=1}^{k_0} B_X\left(p_j, \frac{\varepsilon}{2}\right) \cap X_0\right) \leq \delta.$$

This completes the proof of the claim.  $\square$

According to Claim 3.3, we have  $\text{me}_1(f_n, \tilde{f}) \rightarrow 0$  as  $n \rightarrow \infty$ . This completes the proof of the proposition.  $\square$

Gromov proved in [1, Section 3.1.10] the following proposition by using the distance function  $\text{Tra}_\lambda$  on the space of finite Borel measures. Although the distance function  $\text{Tra}_\lambda$  does not appear in the proof of the following proposition, the proof is essentially the same spirit of his proof.

**Proposition 3.4** (cf. [1, Section 3.1.10]). *Let  $\{\mu_n\}_{n=1}^\infty$  be a sequence of Borel measures on a metric space  $X$  and assume that  $\{\mu_n\}_{n=1}^\infty$  converges weakly to a Borel measure  $\mu$ . Then, we have*

$$\square_1((X, d_X, \mu_n), (X, d_X, \mu)) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

*Proof.* Without loss of generality, we may assume that  $\mu(X) = 1$  and  $\mu_n(X) = 1$  for any  $n \in \mathbb{N}$ . For any  $\varepsilon > 0$ , there exists a sequence  $\{A_i\}_{i=1}^\infty$  of pairwise disjoint Borel subsets of  $X$  satisfying the following properties (1) – (3).

- (1)  $X = \bigcup_{i=1}^\infty A_i = X$ .
- (2) For any  $i \in \mathbb{N}$ ,  $\text{diam } A_i \leq \varepsilon$ .
- (3) For any  $i, n \in \mathbb{N}$ ,  $\mu(\partial A_i) = \mu_n(\partial A_i) = 0$ .

From (1) and (3), there exists  $m \in \mathbb{N}$  such that  $\mu\left(\bigcup_{i=1}^m A_i\right) = \mu\left(\bigcup_{i=1}^m \bar{A}_i\right) > 1 - \varepsilon$ . From the assumption,  $\mu_n(\bar{A}_i) = \mu_n(A_i) \rightarrow \mu(A_i) = \mu(\bar{A}_i)$  as  $n \rightarrow \infty$  for any  $i \in \mathbb{N}$ . Hence, putting

$$\begin{aligned} I_{1n} &:= [0, \mu_n(\bar{A}_1)), \\ I_{in} &:= \left[ \sum_{k=1}^{i-1} \mu_n(\bar{A}_k), \sum_{k=1}^i \mu_n(\bar{A}_k) \right), \quad i = 2, 3, \dots, \\ I_1 &:= [0, \mu(\bar{A}_1)), \\ I_i &:= \left[ \sum_{k=1}^{i-1} \mu(\bar{A}_k), \sum_{k=1}^i \mu(\bar{A}_k) \right), \quad i = 2, 3, \dots, \end{aligned}$$

there exists  $N \in \mathbb{N}$  such that

$$\mathcal{L}(I_{in} \cap I_i) \geq \mu(\bar{A}_i) - \varepsilon/m$$

for any  $n \geq N$  and  $i = 1, 2, \dots, m$ . Fix a parameter  $\phi_i : I_i \rightarrow \bar{A}_i$  of the mm-space  $(\bar{A}_i, d_X, \mu)$  for each  $i = 1, 2, \dots, m$ . For any  $A \subseteq X$ , we indicate by  $\text{Int } A$  its interior. Since  $\mu(\bar{A}_i) = \mu(\text{Int } \bar{A}_i)$ , we have

$$\mu\left(\bigcup_{i=1}^m \text{Int } \bar{A}_i\right) = \sum_{i=1}^m \mu(\bar{A}_i) = \sum_{i=1}^m \mathcal{L}(I_i) = \mathcal{L}\left(\bigcup_{i=1}^m I_i\right).$$

Take a parameter  $\phi : [0, 1] \setminus \bigcup_{i=1}^m I_i \rightarrow X \setminus \bigcup_{i=1}^m \text{Int } \bar{A}_i$  of the mm-space  $(X \setminus \bigcup_{i=1}^m \text{Int } \bar{A}_i, d_X, \mu)$ . Defining a Borel measurable map  $\varphi : [0, 1] \rightarrow X$  by

$$\varphi(t) := \begin{cases} \phi_i(t) & t \in I_i, \quad i = 1, 2, \dots, m, \\ \phi(t) & t \in [0, 1] \setminus \bigcup_{i=1}^m I_i, \end{cases}$$

we see that the map  $\varphi$  is a parameter of  $(X, d_X, \mu)$ . We take any  $n \geq N$ . Take parameters  $\psi_{in} : I_{in} \rightarrow \bar{A}_i$  of  $i = 1, 2, \dots, m$ , of the mm-spaces  $(\bar{A}_i, d_X, \mu_n)$ , and a parameter  $\psi_n : [0, 1] \setminus \bigcup_{i=1}^m I_{in} \rightarrow X \setminus \bigcup_{i=1}^m \text{Int } \bar{A}_i$  of the mm-space  $(X \setminus \bigcup_{i=1}^m \text{Int } \bar{A}_i, d_X, \mu_n)$ . We define a Borel

measurable map  $\varphi_n : [0, 1] \rightarrow X$  by

$$\varphi_n(t) := \begin{cases} \psi_{in}(t) & t \in I_{in}, i = 1, 2, \dots, m, \\ \psi_n(t) & t \in [0, 1] \setminus \bigcup_{i=1}^m I_{im}. \end{cases}$$

The map  $\varphi_n$  is a parameter of the mm-space  $(X, d_X, \mu_n)$  for each  $n \geq N$ . Putting  $B_n := \bigcup_{i=1}^m (I_i \cap I_{in})$ , we have

$$\begin{aligned} \mathcal{L}(B_n) &= \sum_{i=1}^m \mathcal{L}(I_i \cap I_{in}) \geq \sum_{i=1}^m (\mu(A_i) - \varepsilon/m) = \sum_{i=1}^m \mu(A_i) - \varepsilon \\ &= \mu\left(\bigcup_{i=1}^m A_i\right) - \varepsilon \geq 1 - 2\varepsilon. \end{aligned}$$

For any  $s, t \in B_n$ , there exist  $j, k \in \mathbb{N}$  such that  $1 \leq j, k \leq m$ ,  $s \in I_j \cap I_{jn}$ , and  $t \in I_k \cap I_{kn}$ . Since  $\varphi(s), \varphi_n(s) \in \bar{A}_j$ ,  $\varphi(t), \varphi_n(t) \in \bar{A}_k$ , and (2), we have

$$|d_X(\varphi(s), \varphi(t)) - d_X(\varphi_n(s), \varphi_n(t))| \leq d_X(\varphi(s), \varphi_n(s)) + d_X(\varphi(t), \varphi_n(t)) \leq 2\varepsilon.$$

Therefore, we obtain  $\square_1((X, d_X, \mu_n), (X, d_X, \mu)) \leq \square_1(\varphi_n^* d_X, \varphi^* d_X) \leq 2\varepsilon$ . This completes the proof.  $\square$

**Theorem 3.5** (Gromov, cf. [1, Section 3 $\frac{1}{2}$ .15, (3' $_b$ )]).  $\square_1(X_n, X) \rightarrow 0$  as  $n \rightarrow \infty$  if and only if for any  $n \in \mathbb{N}$  there exist a Borel measurable map  $p_n : X_n \rightarrow X$ , a Borel subset  $\tilde{X}_n \subseteq X_n$ , and a positive number  $\varepsilon_n$  satisfying the following conditions (1) – (4).

- (1)  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ .
- (2)  $\mu_{X_n}(X_n \setminus \tilde{X}_n) \leq \varepsilon_n$  for  $n = 1, 2, \dots$ .
- (3)  $|d_{X_n}(x, y) - d_X(p_n(x), p_n(y))| \leq \varepsilon_n$  for any  $x, y \in \tilde{X}_n$ .
- (4) The sequence  $\{(p_n)_*(\mu_{X_n})\}_{n=1}^\infty$  converges weakly to  $\mu_X$ .

*Proof.* Assume that (1) – (4) holds. By virtue of Proposition 3.4, we have  $\square_1(X_n, X) \rightarrow 0$  as  $n \rightarrow \infty$ .

Assume that  $\square_1(X_n, X) \rightarrow 0$  as  $n \rightarrow \infty$ . Without loss of generality, we may assume that  $\mu_X(X) = \mu_{X_n}(X_n) = 1$  for any  $n \in \mathbb{N}$ . From the assumption, there exist parameters  $\varphi : [0, 1] \rightarrow X$  of  $X$  and  $\varphi_n : [0, 1] \rightarrow X_n$  of  $X_n$ ,  $n \in \mathbb{N}$ , such that  $\square_1(\varphi_n^* d_{X_n}, \varphi^* d_X) \rightarrow 0$  as  $n \rightarrow \infty$ . Hence, for each  $n = 1, 2, \dots$ , there exist  $\varepsilon_n > 0$  and compact subset  $K_n \subseteq [0, 1]$  satisfying the following conditions (1)' – (4)':

- (1)'  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ .
- (2)'  $\mathcal{L}(K_n) > 1 - \varepsilon_n$ .
- (3)' For any  $s, t \in K_n$ ,  $|d_X(\varphi(s), \varphi(t)) - d_{X_n}(\varphi_n(s), \varphi_n(t))| < \varepsilon_n$ .
- (4)' The maps  $\varphi|_{K_n} : K_n \rightarrow X$  and  $\varphi_n|_{K_n} : K_n \rightarrow X_n$  are continuous.

By (4)', each set  $\varphi_n(K_n)$  is compact. For each  $n \in \mathbb{N}$ , there exist  $l_n \in \mathbb{N}$  and a sequence  $\{B_{in}\}_{i=1}^{l_n}$  of pairwise disjoint Borel subsets of  $X_n$  such that  $\text{diam } B_{ni} < \varepsilon_n$  for any  $i$  and

$\varphi_n(K_n) = \bigcup_{i=1}^{l_n} B_{in}$ . For each  $i$ , we fix a point  $p_{in} \in B_{in}$ . Then there exist a point  $t_{in} \in K_n$  with  $p_{in} = \varphi_n(t_{in})$ . Put  $q_{in} := \varphi(t_{in}) \in X$ .

**Claim 3.6.**  $\varphi(K_n) \subseteq \bigcup_{i=1}^{l_n} B_X(q_{in}, 2\varepsilon_n)$ .

*Proof.* Take any  $q = \varphi(s) \in \varphi(K_n)$  with  $s \in K_n$ . Since  $\varphi_n(s) \in \varphi_n(K_n) \subseteq \bigcup_{i=1}^l B_{X_n}(p_{in}, \varepsilon_n)$ , there exists  $1 \leq i \leq l_n$  such that  $d_{X_n}(\varphi_n(s), \varphi_n(t_{in})) < \varepsilon_n$ . Hence, by (3)', we have

$$d_X(q, q_{in}) = d_X(\varphi(s), \varphi(t_{in})) < d_{X_n}(\varphi_n(s), \varphi_n(t_{in})) + \varepsilon_n < 2\varepsilon_n.$$

This completes the proof of the claim.  $\square$

We denote by  $\tilde{q}_{1n}, \tilde{q}_{2n}, \dots, \tilde{q}_{m_n n}$  the mutually different elements of  $\{q_{1n}, q_{2n}, \dots, q_{l_n n}\}$ . Put

$$\begin{aligned} C_{1n} &:= \varphi(K_n) \cap B_X(\tilde{q}_{1n}, 2\varepsilon_n) \setminus \{\tilde{q}_{2n}, \tilde{q}_{3n}, \dots, \tilde{q}_{m_n n}\}, \\ C_{in} &:= \varphi(K_n) \cap B_X(\tilde{q}_{in}, 2\varepsilon_n) \setminus \left\{ \bigcup_{j=1}^{i-1} (B_X(\tilde{q}_{jn}, 2\varepsilon_n) \setminus \{\tilde{q}_{in}\}) \cup \{\tilde{q}_{i+1n}, \tilde{q}_{i+2n}, \dots, \tilde{q}_{m_n n}\} \right\}, \\ &\quad i = 2, 3, \dots, m_n. \end{aligned}$$

It is easy to see that  $\tilde{q}_{in} \in C_{in}$ ,  $\varphi(K_n) = \bigcup_{j=1}^{m_n} C_{jn}$ ,  $C_{in} \cap C_{jn} = \emptyset$  for  $i \neq j$ , and  $\text{diam } C_{in} \leq 4\varepsilon_n$ . Take points  $x_n^0 \in X_n$  for any  $n \in \mathbb{N}$  and  $x^0 \in X$ . We define a Borel measurable map  $p_n : X_n \rightarrow X$  by  $p_n(x_n) := q_{in}$  if  $x_n \in B_{in}$  and  $p_n(x_n) := x^0$  if  $x_n \in X_n \setminus \varphi_n(K_n)$ . For each  $i = 1, 2, \dots, m_n$ , we fix  $j$  with  $\tilde{q}_{in} = q_{jn}$  and put  $k_n(i) := j$ .

**Claim 3.7.** *The sequence  $\{(p_n)_*(\mu_{X_n})\}_{n=1}^\infty$  converges weakly to the measure  $\mu_X$ .*

*Proof.* Let  $g : X \rightarrow \mathbb{R}$  be any bounded uniformly continuous function and put  $M := \sup_{x \in X} |g(x)|$ . We shall prove

$$\int_{X_n} (g \circ p_n)(x_n) d\mu_{X_n}(x_n) \rightarrow \int_X g(x) d\mu_X(x) \text{ as } n \rightarrow \infty.$$

Since

$$\begin{aligned} \int_{X_n} (g \circ p_n)(x_n) d\mu_{X_n}(x_n) &= \int_0^1 (g \circ p_n \circ \varphi_n)(s) d\mathcal{L}(s) \\ &= \int_{K_n} (g \circ p_n \circ \varphi_n)(s) d\mathcal{L}(s) + \int_{[0,1] \setminus K_n} (g \circ p_n \circ \varphi_n)(s) d\mathcal{L}(s), \end{aligned}$$

we get

$$\begin{aligned} & \left| \int_{X_n} (g \circ p_n)(x_n) d\mu_{X_n}(x_n) - \int_{K_n} (g \circ p_n \circ \varphi_n)(s) d\mathcal{L}(s) \right| \\ & \leq \int_{[0,1] \setminus K_n} |(g \circ p_n \circ \varphi_n)(s)| d\mathcal{L}(s) \leq M\varepsilon_n. \end{aligned}$$

Similary, we have

$$\left| \int_X g(x) d\mu_X(x) - \int_{K_n} (g \circ \varphi)(s) d\mathcal{L}(s) \right| \leq M\varepsilon_n.$$

Since for any  $s \in \varphi_n^{-1}(B_{in}) \cap \varphi^{-1}(C_{jn})$

$$d_{X_n}(\varphi_n(s), p_{in}) \leq \varepsilon_n \text{ and } d_X(\varphi(s), \tilde{q}_{jn}) \leq 2\varepsilon_n,$$

we obtain

$$\begin{aligned} & d_X(\varphi(s), q_{in}) \\ & \leq d_X(\varphi(s), \tilde{q}_{jn}) + |d_X(\tilde{q}_{jn}, q_{in}) - d_{X_n}(\varphi_n(s), p_{in})| + d_{X_n}(\varphi_n(s), p_{in}) \\ & < 2\varepsilon_n + |d_X(\tilde{q}_{jn}, q_{in}) - d_{X_n}(p_{k_n(j)n}, p_{in})| + d_{X_n}(p_{k_n(j)n}, \varphi_n(s)) + \varepsilon_n \\ & < 4\varepsilon_n + d_{X_n}(p_{k_n(j)n}, \varphi_n(s)) \\ & < 4\varepsilon_n + d_X(\tilde{q}_{jn}, \varphi(s)) + \varepsilon_n \leq 7\varepsilon_n. \end{aligned}$$

Since  $g$  is uniformly continuous function on  $X$ , for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|g(x) - g(y)| < \varepsilon$  for any  $x, y \in X$  with  $d_X(x, y) < \delta$ . Hence for any  $n \in \mathbb{N}$  with  $7\varepsilon_n < \delta$ , we have  $|g(q_{in}) - g(\varphi(s))| < \varepsilon$ , which implies that

$$\begin{aligned} & \left| \int_{K_n} (g \circ p_n \circ \varphi_n)(s) d\mathcal{L}(s) - \int_{K_n} (g \circ \varphi)(s) d\mathcal{L}(s) \right| \\ & \leq \sum_{i,j=1}^{l_n, m_n} \int_{\varphi_n^{-1}(B_{in}) \cap \varphi^{-1}(C_{jn}) \cap K_n} |g(q_{in}) - g(\varphi(s))| d\mathcal{L}(s) \\ & < \varepsilon \sum_{i,j=1}^{l_n, m_n} \mathcal{L}(\varphi_n^{-1}(B_{in}) \cap \varphi^{-1}(C_{jn}) \cap K_n) = \varepsilon \mathcal{L}(K_n) \leq \varepsilon. \end{aligned}$$



Therefore, we obtain

$$\begin{aligned}
 & \left| \int_{X_n} (g \circ p_n)(x_n) d\mu_{X_n}(x_n) - \int_X g(x) d\mu_X(x) \right| \\
 & \leq \left| \int_{X_n} (g \circ p_n)(x_n) d\mu_{X_n}(x_n) - \int_{K_n} (g \circ p_n \circ \varphi_n)(s) d\mathcal{L}(s) \right| \\
 & \quad + \left| \int_{K_n} (g \circ p_n \circ \varphi_n)(s) d\mathcal{L}(s) - \int_{K_n} (g \circ \varphi)(s) d\mathcal{L}(s) \right| \\
 & \quad + \left| \int_{K_n} (g \circ \varphi)(s) d\mathcal{L}(s) - \int_X g(x) d\mu_X(x) \right| \\
 & \leq 2M\varepsilon_n + \varepsilon.
 \end{aligned}$$

This completes the proof of the claim.  $\square$

For any  $x \in B_{in}, y \in B_{jn}$ , we obtain

$$\begin{aligned}
 & |d_{X_n}(x, y) - d_X(p_n(x), p_n(y))| \\
 & = |d_{X_n}(x, y) - d_X(q_{in}, q_{jn})| \\
 & \leq |d_{X_n}(x, y) - d_{X_n}(p_{in}, p_{jn})| + |d_{X_n}(p_{in}, p_{jn}) - d_X(q_{in}, q_{jn})| \\
 & \leq d_{X_n}(x, p_{in}) + d_{X_n}(y, p_{jn}) + |d_{X_n}(p_{in}, p_{jn}) - d_X(q_{in}, q_{jn})| \\
 & < 2\varepsilon_n + \varepsilon_n \\
 & = 3\varepsilon_n.
 \end{aligned}$$

Therefore, we have complete the proof of Theorem 3.5.  $\square$

Modifying the proof of Theorem 3.5, we get the following corollary:

**Corollary 3.8.** *Let  $X$  and  $X_n$ ,  $n \in \mathbb{N}$ , be compact mm-spaces. Assume that  $X = \text{Supp } \mu_X$ ,  $X_n = \text{Supp } \mu_{X_n}$ , and  $\mu_X(X) = \mu_{X_n}(X_n)$  for any  $n \in \mathbb{N}$ . Then, the sequence  $\{X_n\}_{n=1}^\infty$  converges to  $X$  with respect to  $\square_0$  if and only if  $\{X_n\}_{n=1}^\infty$  converges to  $X$  in the sense of the measured Gromov-Hausdorff convergence.*

Combining Proposition 3.1 and Theorem 3.5, we get the following corollary:

**Corollary 3.9.** *Assume that  $\square_\lambda(X, Y) = 0$ . Then, two mm-spaces  $X$  and  $Y$  are isomorphic to each other.*

#### 4. STABILITY OF HOMOGENEITY

We say that an mm-space  $X$  *Lipschitz dominates* an mm-space  $Y$  and write  $X \succ Y$  if there exist 1-Lipschitz map  $p : \text{Supp } \mu_X \rightarrow \text{Supp } \mu_Y$  and  $c \geq 1$  such that  $p_*(\mu_X) = c\mu_Y$ .

**Theorem 4.1** (Gromov, cf. [1, Section 3 $\frac{1}{2}$ .15, (b)]). *Assume that  $\square_\lambda(X_n, X), \square_\lambda(Y_n, Y) \rightarrow 0$  as  $n \rightarrow \infty$  and  $X_n \succ Y_n$  for any  $n \in \mathbb{N}$ . Then we have  $X \succ Y$ .*

*Proof.* Without loss of generality, we may assume that  $\mu_{X_n}(X_n) = \mu_{Y_n}(Y_n) = \mu_X(X) = \mu_Y(Y) = 1$ ,  $X = \text{Supp } \mu_X$ ,  $Y = \text{Supp } \mu_Y$ ,  $X_n = \text{Supp } \mu_{X_n}$ , and  $Y_n = \text{Supp } \mu_{Y_n}$  for any  $n \in \mathbb{N}$ . From the assumption, for any  $n \in \mathbb{N}$  there exists a 1-Lipschitz map  $f_n : X_n \rightarrow Y_n$  such that  $(f_n)_*(\mu_{X_n}) = \mu_{Y_n}$ . By using Theorem 3.5, for any  $n \in \mathbb{N}$  there exists a Borel measurable map  $q_n : Y_n \rightarrow Y$ , a compact subset  $\tilde{Y}_n \subseteq Y_n$ , and  $\varepsilon_n > 0$  such that

- (1)  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ ,
- (2)  $\mu_{Y_n}(Y_n \setminus \tilde{Y}_n) \leq \varepsilon_n$  for  $n = 1, 2, \dots$ ,
- (3)  $|d_{Y_n}(x, y) - d_Y(q_n(x), q_n(y))| \leq \varepsilon_n$  for any  $x, y \in \tilde{Y}_n$ ,
- (4) The sequence  $\{(q_n)_*(\mu_{Y_n})\}_{n=1}^\infty$  converges weakly to  $\mu_Y$ .

From now on, we define a Borel measurable map  $p_n : X \rightarrow X_n$  as follows: Since  $\square_1(X_n, X) \rightarrow 0$  as  $n \rightarrow \infty$ , there exists a parameter  $\varphi_n : [0, 1] \rightarrow X_n$  and  $\varphi : [0, 1] \rightarrow X$  such that  $\square_1(\varphi_n^* d_{X_n}, \varphi^* d_X) \rightarrow 0$  as  $n \rightarrow \infty$ . Hence, there exists a compact subset  $K_n \subseteq [0, 1]$  and  $\varepsilon'_n > 0$  satisfying the following properties (1)' – (4)':

- (1)'  $\varepsilon'_n \rightarrow 0$  as  $n \rightarrow \infty$ .
- (2)'  $\mathcal{L}(K_n) > 1 - \varepsilon'_n$ .
- (3)' For any  $s, t \in K_n$ ,  $|d_{X_n}(\varphi_n(s), \varphi_n(t)) - d_X(\varphi(s), \varphi(t))| < \varepsilon'_n$ .
- (4)' The maps  $\varphi_n|_{K_n} : K_n \rightarrow [0, 1]$  and  $\varphi_n|_{K_n} : K_n \rightarrow [0, 1]$  are continuous.

By (4)', the sets  $\varphi_n(K_n) \cap f_n^{-1}(\tilde{Y}_n)$  and  $\tilde{X}_n := \varphi(K_n \cap \varphi_n^{-1}(f_n^{-1}(\tilde{Y}_n)))$  are compact. For each  $n \in \mathbb{N}$ , there exist  $l_n \in \mathbb{N}$  and a sequence  $\{C_{in}\}_{i=1}^{l_n}$  of pairwise disjoint Borel subsets of  $X$  such that  $\text{diam } C_{in} < \varepsilon_n$  for any  $i$  and  $\tilde{X}_n = \bigcup_{i=1}^{l_n} C_{in}$ . For each  $i$ , we fix a point  $q_{in} \in C_{in}$ . Then there exist a point  $t_{in} \in K_n \cap \varphi_n^{-1}(f_n^{-1}(\tilde{Y}_n))$  with  $q_{in} = \varphi(t_{in})$ . Put  $p_{in} := \varphi(t_{in}) \in X$ . Then, we get  $\varphi_n(K_n) \cap f_n^{-1}(\tilde{Y}_n) \subseteq \bigcup_{i=1}^{l_n} B_{X_n}(p_{in}, 2\varepsilon'_n)$ . We denote by  $\tilde{p}_{1n}, \tilde{p}_{2n}, \dots, \tilde{p}_{m_n n}$  the mutually different elements of  $\{p_{1n}, p_{2n}, \dots, p_{l_n n}\}$ . Put

$$B_{1n} := \varphi_n(K_n) \cap f_n^{-1}(\tilde{Y}_n) \cap B_{X_n}(\tilde{p}_{1n}, 2\varepsilon'_n) \setminus \{\tilde{p}_{2n}, \tilde{p}_{3n}, \dots, \tilde{p}_{m_n n}\},$$

$$B_{in} := \varphi_n(K_n) \cap f_n^{-1}(\tilde{Y}_n) \cap B_{X_n}(\tilde{p}_{in}, 2\varepsilon'_n)$$

$$\setminus \left\{ \bigcup_{j=1}^{i-1} (B_{X_n}(\tilde{p}_{jn}, 2\varepsilon'_n) \setminus \{\tilde{p}_{in}\}) \cup \{\tilde{p}_{i+1n}, \tilde{p}_{i+2n}, \dots, \tilde{p}_{m_n n}\} \right\}, i = 2, 3, \dots, m_n.$$

It is easy to see that  $\tilde{p}_{in} \in B_{in}$ ,  $\varphi_n(K_n) \cap f_n^{-1}(\tilde{Y}_n) = \bigcup_{j=1}^{m_n} B_{jn}$ ,  $B_{in} \cap B_{jn} = \emptyset$  for  $i \neq j$ , and  $\text{diam } B_{in} \leq 4\varepsilon'_n$ . Take points  $x_n^0 \in X_n$  for any  $n \in \mathbb{N}$  and  $x^0 \in X$ . We put  $p_n(x) := p_{in}$  if  $x \in C_{in}$  and  $p_n(x) := x^0$  if  $x \in X \setminus \tilde{X}_n$ . The same proof in Theorem 3.5 implies the following: There exists a positive number  $\delta_n > 0$  such that  $\delta_n \rightarrow 0$  as  $n \rightarrow \infty$ ,  $\mu_X(X \setminus \tilde{X}_n) < \delta_n$ , and  $|d_{X_n}(p_n(x), p_n(x')) - d_X(x, x')| < \delta_n$  for any  $x, x' \in \tilde{X}_n$ .

Put  $g_n := q_n \circ f_n \circ p_n : X \rightarrow Y$ . For any  $x, x' \in \tilde{X}_n$ ,

$$\begin{aligned}
d_Y(g_n(x), g_n(x')) - d_X(x, x') &\leq |d_Y(g_n(x), g_n(x')) - d_{Y_n}((f_n \circ p_n)(x), (f_n \circ p_n)(x'))| \\
&\quad + d_{Y_n}((f_n \circ p_n)(x), (f_n \circ p_n)(x')) - d_X(x, x') \\
&\leq \varepsilon_n + (d_{X_n}(p_n(x), p_n(x')) - d_X(x, x')) \\
&\quad + (d_{Y_n}((f_n \circ p_n)(x), (f_n \circ p_n)(x')) - d_{X_n}(p_n(x), p_n(x'))) \\
&\leq \varepsilon_n + \delta_n.
\end{aligned}$$

Hence,  $g_n$  is a 1-Lipschitz up to  $(\varepsilon_n + \delta_n)$  Borel measurable map.

**Claim 4.2.** *The sequence  $\{(g_n)_*(\mu_X)\}_{n=1}^\infty$  converges weakly to the measure  $\mu_Y$ .*

*Proof.* Let  $h : Y \rightarrow \mathbb{R}$  be any bounded uniformly continuous function on  $Y$ . We will prove that

$$\int_X (h \circ g_n)(x) d\mu_X(x) \rightarrow \int_Y h(y) d\mu_Y(y) \text{ as } n \rightarrow \infty.$$

Since

$$\int_{X_n} (h \circ q_n \circ f_n)(x_n) d\mu_{X_n}(x_n) = \int_{Y_n} (h \circ q_n)(y_n) d\mu_{Y_n}(y_n) \rightarrow \int_Y h d\mu_Y(y) \text{ as } n \rightarrow \infty,$$

it suffices to prove that

$$\left| \int_X (h \circ g_n)(x) d\mu_X(x) - \int_{X_n} (h \circ q_n \circ f_n)(x_n) d\mu_{X_n}(x_n) \right| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Put  $M := \sup_{y \in Y} |h(y)|$ . Take any  $\varepsilon > 0$ . We have

$$\begin{aligned}
&\left| \int_X (h \circ g_n)(x) d\mu_X(x) - \int_{K_n \cap \varphi_n^{-1}(f_n^{-1}(\tilde{Y}_n))} (h \circ g_n \circ \varphi)(s) d\mathcal{L}(s) \right| \\
&\leq \int_{[0,1] \setminus (K_n \cap \varphi_n^{-1}(f_n^{-1}(\tilde{Y}_n)))} M d\mathcal{L}(s) < M\varepsilon
\end{aligned}$$

and

$$\begin{aligned}
&\left| \int_{X_n} (h \circ q_n \circ f_n)(x_n) d\mu_{X_n}(x_n) - \int_{K_n \cap \varphi_n^{-1}(f_n^{-1}(\tilde{Y}_n))} (h \circ q_n \circ f_n \circ \varphi_n)(s) d\mathcal{L}(s) \right| \\
&\leq \int_{[0,1] \setminus (K_n \cap \varphi_n^{-1}(f_n^{-1}(\tilde{Y}_n)))} M d\mathcal{L}(s) < M\varepsilon
\end{aligned}$$

for any sufficiently large  $n \in \mathbb{N}$ . For any  $\delta > 0$ , we put

$$\rho_h(\delta) := \sup\{|h(u) - h(v)| \mid d_Y(u, v) < \delta, u, v \in Y\}.$$

Let  $\varepsilon' > 0$  with  $\rho_h(2\varepsilon') < \varepsilon$ . For any  $s \in K_n \cap \varphi_n^{-1}(f_n^{-1}(\tilde{Y}_n)) \cap \varphi^{-1}(C_{kn}) \cap \varphi_n^{-1}(B_{jn})$ , we get  $d_{X_n}(\varphi_n(s), p_{kn}) < \varepsilon'$  for sufficiently large  $n \in \mathbb{N}$  by the same method of the proof in

Theorem 3.5. Assume that  $x, y \in f_n^{-1}(\tilde{Y}_n)$  and  $d_{X_n}(x, y) < \varepsilon'$ . Then, for any sufficiently large  $n \in \mathbb{N}$ , we have

$$d_Y((q_n \circ f_n)(x), (q_n \circ f_n)(y)) \leq d_{Y_n}(f_n(x), f_n(y)) + \varepsilon_n \leq d_X(x, y) + \varepsilon_n < 2\varepsilon'.$$

Hence, we get

$$\begin{aligned} & \left| \int_{K_n \cap \varphi_n^{-1}(f_n^{-1}(\tilde{Y}_n))} \left( (h \circ q_n \circ f_n)(p_n(\varphi(s))) - (h \circ q_n \circ f_n)(\varphi_n(s)) \right) d\mathcal{L}(s) \right| \\ & \leq \sum_{k,j=1}^{l_n, m_n} \int_{K_n \cap \varphi_n^{-1}(f_n^{-1}(\tilde{Y}_n)) \cap \varphi^{-1}(C_{kn}) \cap \varphi_n^{-1}(B_{jn})} \left| (h \circ q_n \circ f_n)(p_{kn}) - (h \circ q_n \circ f_n)(\varphi_n(s)) \right| d\mathcal{L}(s) \\ & \leq \sum_{k,j=1}^{l_n, m_n} \int_{K_n \cap \varphi_n^{-1}(f_n^{-1}(\tilde{Y}_n)) \cap \varphi^{-1}(C_{kn}) \cap \varphi_n^{-1}(B_{jn})} \rho_h(2\varepsilon') d\mathcal{L}(s) \leq \varepsilon. \end{aligned}$$

This completes the proof of the claim.  $\square$

Combining Proposition 3.1 and Claim 4.2, we may assume that the sequence  $\{g_n\}_{n=1}^\infty$  converges with respect to the distance function  $\text{me}_1$ . Let  $g : X \rightarrow Y$  be its limit. Then this  $g$  is obviously a 1-Lipschitz map.

**Claim 4.3.** *The sequence  $\{(g_n)_*(\mu_{X_n})\}_{n=1}^\infty$  converges weakly to the measure  $g_*(\mu_X)$ .*

*Proof.* Let  $U \subseteq Y$  be any open subset. Put  $U(\delta) := \{y \in U \mid d_Y(y, X \setminus U) > \delta\}$  for any  $\delta > 0$ . For any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $\mu_X(f^{-1}(U)) < \mu_X(f^{-1}(U(\delta))) + \varepsilon$ . Therefore, we obtain

$$\begin{aligned} \mu_X(f^{-1}(U)) & < \mu_X(f^{-1}(U(\delta))) \\ & = \limsup_{n \rightarrow \infty} \mu_X(f^{-1}(U(\delta)) \cap \{x \in X \mid d_Y(f_n(x), f(x)) < \delta\}) \\ & \leq \liminf_{n \rightarrow \infty} \mu_X(f_n^{-1}(U)). \end{aligned}$$

This completes the proof of the claim.  $\square$

Combining Claim 4.2 and Claim 4.3, we get  $g_*(\mu_X) = \mu_Y$ . This completes the proof of the theorem.  $\square$

Modifying the proof of Theorem 4.1, we get the following corollary:

**Corollary 4.4.** *Assume that a sequence  $\{M_n\}_{n=1}^\infty$  of compact homogeneous Riemannian manifolds convergence to an mm-space  $X$  with respect to the distance function  $\square_\lambda$  and  $X = \text{Supp } \mu_X$ . Then, the limit space  $X$  is also homogeneous and every isometry  $g : X \rightarrow X$  satisfy  $g_*(\mu_X) = \mu_X$ .*

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